DP IB Maths: AA SL

Your notes

1.5 Binomial Theorem

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Binomial Theorem

What is the Binomial Theorem?

- The **binomial theorem (**sometimes known as the binomial expansion) gives a method for expanding a **two-term** expression in a bracket raised to a power
  - A **binomial expression** is in fact any two terms inside the bracket, however in IB the expression will usually be linear
- To expand a bracket with a two-term expression in:
  - First choose the most appropriate parts of the expression to assign to *a* and *b*
  - Then use the formula for the binomial theorem:

$$(a+b)^n = a^n + {}^nC_1\, a^{n-1}\, b + \ldots + {}^nC_r\, a^{n-r}\, b^r + \ldots + b^n$$

  - where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$
    - See below for more information on ${}^nC_r$
    - You may also see ${}^nC_r$ written as $\binom{n}{r}$ or ${}_nC_r$

- You will usually be asked to find the first three or four terms of an expansion
- Look out for whether you should give your answer in **ascending** or **descending** powers of *x*
  - For **ascending** powers start with the constant term, a^n
  - For **descending** powers start with the term with *x* in
    - You may wish to swap *a* and *b* over so that you can follow the general formula given in the formula book
- If you are not writing the full expansion you can either
  - show that the sequence continues by putting an ellipsis (...) after your final term
  - or show that the terms you have found are an approximation of the full sequence by using the sign for approximately equals to (≈)

How do I find the coefficient of a single term?

- Most of the time you will be asked to find the coefficient of a term, rather than carry out the whole expansion
- Use the formula for the general term

$${}^nC_r\, a^{n-r}\, b^r$$

- The question will give you the power of *x* of the term you are looking for

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- Use this to choose which value of r you will need to use in the formula
- This will depend on where the x is in the bracket
- The laws of indices can help you decide which value of r to use:
  - For $(a + bx)^n$ to find the coefficient of x^r use $a^{n-r}(bx)^r$
  - For $(a + bx^2)^n$ to find the coefficient of x^r use $a^{n-\frac{r}{2}}(bx^2)^{\frac{r}{2}}$
  - For $\left(a + \frac{b}{x}\right)^n$ look at how the powers will cancel out to decide which value of r to use
  - So for $\left(3x + \frac{2}{x}\right)^8$ to find the coefficient of x^2 use the term with $r = 3$ and to find the constant term use the term with $r = 4$
  - There are a lot of variations of this so it is usually easier to see this by inspection of the exponents
- You may also be given the coefficient of a particular term and asked to find an unknown in the brackets
  - Use the laws of indices to choose the correct term and then use the binomial theorem formula to form and solve and equation

Examiner Tip

- Binomial expansion questions can get messy, use separate lines to keep your working clear and always put terms in brackets

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Worked example

Find the first three terms, in ascending powers of x, in the expansion of $(3-2x)^5$.

$a = 3 \quad b = -2x \quad n = 5$

Substitute values into the formula for $(a+b)^n$

$(a+b)^n = a^n + {}^nC_1 a^{n-1} b + \ldots + {}^nC_r a^{n-r} b^r + \ldots + b^n$

Question asks for ascending powers of x, so start with the constant term, a^n.

$(3-2x)^5 = 3^5 + {}^5C_1 (3)^{5-1}(-2x) + {}^5C_2 (3)^{5-2}(-2x)^2 + \ldots$

Watch out for the negative

$\approx 243 + 5 \times 81 \times -2x + 10 \times 27 \times 4x^2$

$\approx 243 - 810x + 1080x^2$

$(3-2x)^5 \approx 243 - 810x + 1080x^2$

The Binomial Coefficient nCr

What is ${}^{n}C_{r}$?

- If we want to find the number of ways to **choose** *r* items out of *n* different objects we can use the formula for ${}^{n}C_{r}$
  - The formula for *r* **combinations** of *n* items is ${}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$
  - This formula is given in the formula booklet along with the formula for the binomial theorem
  - The function ${}^{n}C_{r}$ can be written $\begin{pmatrix} n \\ r \end{pmatrix}$ or ${}_{n}C_{r}$ and is often read as '*n choose r*'
    - Make sure you can find and use the button on your GDC

How does ${}^{n}C_{r}$ relate to the binomial theorem?

- The formula ${}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$ is also known as a **binomial** coefficient
- For a binomial expansion $(a\ +\ b)^{n}$ the coefficients of each term will be ${}^{n}C_{0}$, ${}^{n}C_{1}$ and so on up to ${}^{n}C_{n}$
  - The coefficient of the r^{th} term will be ${}^{n}C_{r}$
- ${}^{n}C_{n}\ =\ {}^{n}C_{0}\ =\ 1$
- The binomial coefficients are symmetrical, so ${}^{n}C_{r}\ =\ {}^{n}C_{n\ -\ r}$
  - This can be seen by considering the formula for ${}^{n}C_{r}$
  - ${}^{n}C_{n-r} = \dfrac{n!}{(n-r)!(n-(n-r))!} = \dfrac{n!}{r!(n-r)!} = nC_{r}$

Examiner Tip

- You will most likely need to use the formula for nCr at some point in your exam
  - Practice using it and don't always rely on your GDC
  - Make sure you can find it easily in the formula booklet

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Worked example

Without using a calculator, find the coefficient of the term in x^3 in the expansion of $(1 + x)^9$.

$n = 9, \quad a = 1, \quad b = x$

Substitute values into the formula for the binomial theorem:

$(a+b)^n = a^n + \ldots + {}^nC_r a^{n-r} b^r + \ldots + b^n$

where $\quad {}^nC_r = \frac{n!}{r!(n-r)!}$

$(1+x)^9 = \sum_{r=0}^{9} {}^9C_r (1)^{9-r} (x)^r$ ← Coefficient of x^3 occurs when $r = 3$.

$r = 3$ gives $\quad {}^9C_3 \times (1)^{9-3} (x)^3$

Non-calculator, so work out ${}^nC_r$ separately:

$${}^9C_3 = \frac{9!}{3!(9-3)!} = \frac{9 \times 8 \times 7 \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2}}{(3 \times 2)(\cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2})}$$

$$= \frac{9 \times 8 \times 7}{6} = 84$$

so the term when $r = 3$ is $84 \times (1)^6 \times x^3$

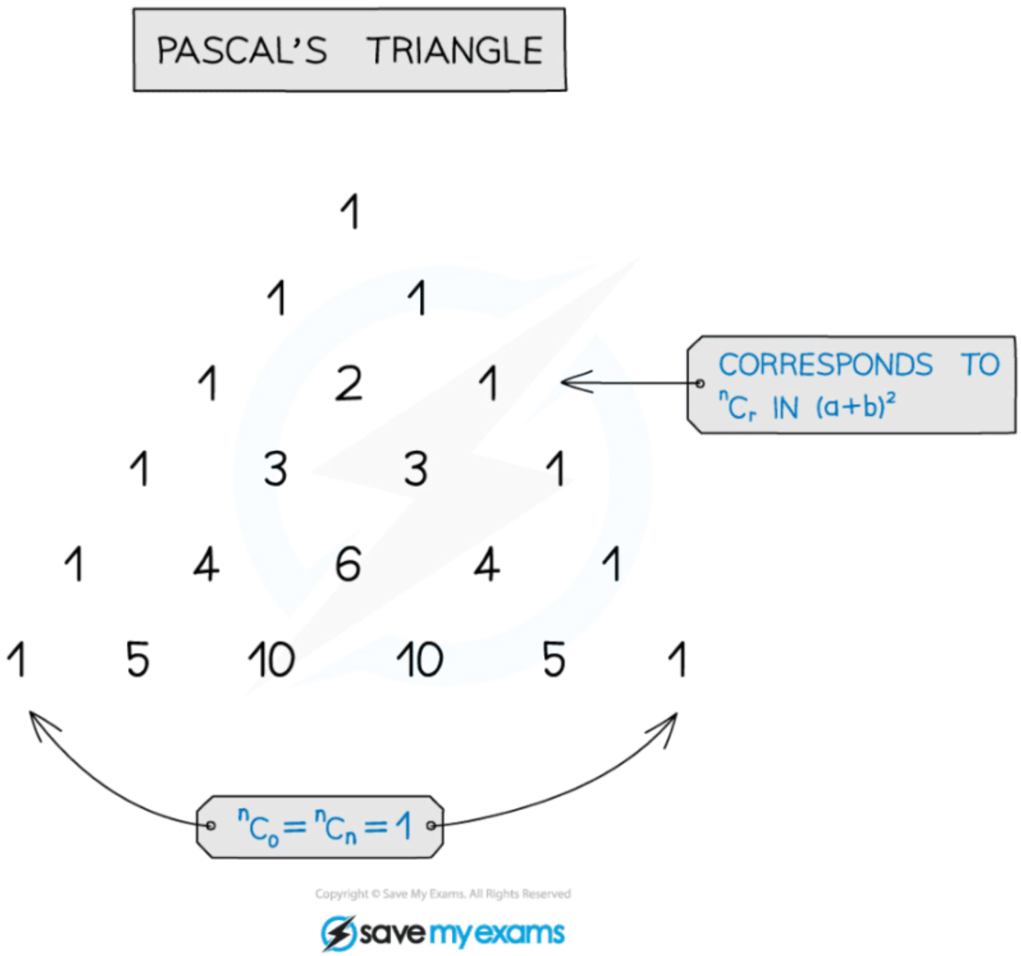
$= 84x^3$

Coefficient of x^3 = 84

Pascal's Triangle

What is Pascal's Triangle?

- Pascal's triangle is a way of arranging the binomial coefficients and neatly shows how they are formed
  - Each term is formed by adding the two terms above it
  - The first row has just the number 1
  - Each row begins and ends with a number 1
  - From the third row the terms in between the 1s are the sum of the two terms above it



How does Pascal's Triangle relate to the binomial theorem?

- Pascal's triangle is an alternative way of finding the binomial coefficients, ^nC_r
  - It can be useful for finding for smaller values of n without a calculator

- However for larger values of n it is slow and prone to arithmetic errors
- Taking the first row as zero, $\left({}^0C_0 = 1\right)$, each row corresponds to the n^{th} row and the term within that row corresponds to the r^{th} term

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Examiner Tip

- In the non-calculator exam Pascal's triangle can be helpful if you need to get the coefficients of an expansion quickly, provided the value of *n* is not too big

Worked example

Write out the 7th row of Pascal's triangle and use it to find the value of ${}^6C_4$.